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# On the Eigenvalues of an Integral Equation Arising in the Theory of Band-Limited Signals\*

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## INTRODUCTION

This paper is concerned with the asymptotic behavior as  $c \rightarrow \infty$  of the eigenvalues  $\lambda$  of the integral equation

$$\lambda \phi(y) = \frac{1}{\pi} \int_{-c}^c \phi(x) \frac{\sin(x-y)}{x-y} dx \quad (|y| < c). \quad (1)$$

It will be convenient to change (1) by the substitutions

$$c = a^2, \quad x = as, \quad y = at, \quad \phi(x) = f(s)$$

to

$$\lambda f(t) = \frac{1}{\pi} \int_{-a}^a f(s) \frac{\sin a(s-t)}{s-t} ds. \quad (2)$$

This equation arises in the following problem: Let  $F \in L^2(-a, a)$ ,

$$\int_{-a}^a |F(s)|^2 ds = 1, \quad (3)$$

put

$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-a}^a F(s) e^{ist} ds$$

( $G$  is a "band-limited signal"). Then direct calculation gives

$$\int_{-a}^a |G(t)|^2 dt = \frac{1}{\pi} \int_{-a}^a \int_{-a}^a \overline{F(u)} F(s) \frac{\sin a(s-u)}{s-u} du ds. \quad (4)$$

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A well-known variational principle says that the maximum of the expression (4) for  $F$  varying in  $L^2(-a, a)$  subject to (3) is attained when  $F$  is equal to an eigenfunction of (2) belonging to the largest eigenvalue  $\lambda_0$  of (2), and further

$$\lambda_0 = \max \int_{-a}^a |G(t)|^2 dt.$$

(The existence of eigenvalues and eigenfunctions of (2) is assured by the general theory of integral equations with symmetric  $L^2$ -kernel.)

The problem of maximising (4) under the condition (3) is of some engineering interest and has therefore been treated by several authors [1-4]. In particular, it was found that the eigenfunctions of (2) are closely related to a known system of special functions, the prolate spheroidal wave functions. A summary of the known results which will be required later is given in Theorem A of Section I.

An investigation of the eigenvalues of (1) is of interest also, because (1) is a convolution integral equation of the form

$$\lambda \phi(y) = \int_{-c}^c \phi(x) k(x-y) dx. \quad (5)$$

Equations of this type, and in particular the behavior of their eigenvalues as  $c \rightarrow \infty$  have been intensively studied by H. Widom, S. Parter, and many others (see [5] and literature cited there). In all of this work it is assumed that  $k(u) \in L(-\infty, \infty)$  and that the Fourier transform  $K(u)$  of  $k(u)$  has a dominant maximum at  $v = 0$ , where

$$K(v) = K(0) - b |v|^\alpha + o(|v|^\alpha) \quad (\alpha > 0)$$

as  $v \rightarrow 0$ . Under these hypotheses the asymptotic behavior of the  $m$ th eigenvalue of (5) is given by

$$\lambda_m = K(0) - \beta_m c^{-\alpha} + o(c^{-\alpha}) \quad (c \rightarrow \infty), \quad (6)$$

where  $\beta_m$  is independent of  $c$ . For the equation (1) the hypotheses which were mentioned above are not satisfied:

$$k(u) = \sin u/\pi u$$

is not in  $L(-\infty, \infty)$  and

$$K(v) = \int k(u) e^{iuv} du = \begin{cases} 1 & (|v| < 1) \\ 0 & (|v| > 1) \end{cases}.$$

The eigenvalues of (1) show a behavior which is quite different from the behavior described by (6). I shall prove

**THEOREM 1.** *Let  $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots$  be the eigenvalues of the integral equation (1). Then*

$$1 - \lambda_n \sim 4\pi^{1/2} 8^n (n!)^{-1} a^{2n+1} e^{-2a^2} \quad a \rightarrow \infty; \quad a^2 = c.$$

Throughout the proof the letter  $K$  will stand for a positive number independent of  $a$ , but possibly depending on  $n$ . The value of  $K$  need not be the same at every occurrence.

#### I. SUMMARY OF KNOWN RESULTS AND THE DIFFERENTIAL EQUATION FOR $\lambda_n$ .

The following theorem summarises known results about the eigenvalues and eigenfunctions of the integral equation (2).

**THEOREM A.** *The eigenvalues of (2) form a denumerable set  $\lambda_0, \lambda_1, \cdots$ . They satisfy*

$$1 > \lambda_0 > \lambda_1 > \lambda_2 \cdots > 0. \quad (1.1)$$

*To each  $\lambda_j$  belongs a real-valued eigenfunction  $f_j(t)$ , unique up to a factor  $\pm 1$  and such that*

$$\int_{-a}^a f_j(t) f_k(t) dt = \delta_{jk} \quad (j, k = 0, 1, 2, \cdots). \quad (1.2)$$

*The function  $f_j(t)$  is even, if  $j$  is even, odd if  $j$  is odd.*

*The function  $f_n(t)$  also satisfies*

$$i^n \mu_n f_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-a}^a f_n(s) e^{ist} ds, \quad (1.3)$$

where

$$\mu_n^2 = \lambda_n.$$

*By means of (1.3)  $f_n(t)$  is defined as an entire function of the complex variable  $t$ .*

*Write  $L$  for the differential operator*

$$L y(t) = \frac{d}{dt} \left\{ (t^2 - a^2) \frac{dy}{dt} \right\} + a^2 t^2 y. \quad (1.4)$$

*The eigenvalue problem*

$$Ly = \chi y \quad (-\infty < t < \infty), \quad (1.5)$$

$y(t)$  continuous for all real  $t$ ,

has eigenvalues

$$0 < \chi_0 < \chi_1 < \dots.$$

*The corresponding eigenfunctions are  $f_0(t), f_1(t), \dots$  respectively.*

*As  $a \rightarrow \infty$*

$$\chi_n = (2n + 1) a^2 + O(1) \quad (n = 0, 1, 2, \dots). \quad (1.6)$$

*If the parameter  $a$  is allowed to vary, then  $f_n(t) = f_n(t, a)$  is a continuous function of  $t$  and  $a$  in  $-\infty < t < \infty, a > 1$ .*

All the statements of Theorem A with the exception of (1.6) and the final remark are proved in [3]. It is also shown there that  $f_n(t)$  is expressible in terms of prolate spheroidal wave functions. In the notation of [6]

$$f_n(t) = f_n(t, a) = \left( \frac{2n+1}{2a} \right)^{1/2} p_{s_n} \left( \frac{t}{a}; a^4 \right), \quad (1.7)$$

$$\chi_n = \chi_n(a) = \lambda_n(a^4) + a^4. \quad (1.8)$$

The last remark of Theorem A follows at once from the stronger statement that  $p_{s_n}(u; \gamma^2)$  is an entire function of the two complex variables  $u$  and  $\gamma^2$  [6, p. 221]. And (1.6) follows from the asymptotic development of  $\lambda_n(a^4)$  given in [6], Chap. 3, Theorem 9, p. 243] (here  $\gamma = a^2, m = 0$ ). The same theorem also yields the asymptotic expansions

$$f_n(t, a) = (2^n n! \pi^{1/2})^{-1/2} \phi_n(t) + O(a^{-3/2}) \quad (a \rightarrow \infty), \quad (1.9)$$

$$f'_n(t, a) = (2^n n! \pi^{1/2})^{-1/2} \phi'_n(t) + O(a^{-1}) \quad (a \rightarrow \infty), \quad (1.10)$$

uniformly in  $-a \leq t \leq a$ . Here

$$\begin{aligned} \phi_n(t) &= (-1)^n e^{t^2/2} \left( \frac{d}{dt} \right)^n e^{-t^2} \\ &= H_n(t) e^{-t^2/2} \\ &= (2^n t^n + \dots) e^{-t^2/2}, \end{aligned} \quad (1.11)$$

where  $H_n(t)$  is the  $n$ th Hermite polynomial.

The principal idea of the proof of Theorem 1 is contained in

LEMMA 1. *The  $n$ th eigenvalue of (2), considered as function of  $a$  satisfies*

$$\frac{d\lambda_n}{da} = 4\lambda_n f_n^2(a) = 4\lambda_n f_n^2(a, a). \quad (1.12)$$

PROOF. Where it seems necessary the dependence on the parameter  $a$  will be emphasized by writing  $f_n(t, a)$ ,  $\mu_n(a) \cdots$  in place of  $f_n(t)$ ,  $\mu_n \cdots$ .

Let  $b > a$ . Multiply (1.3) by  $f_n(t, b)$  and integrate from  $-b$  to  $b$ :

$$\begin{aligned} i^n \mu_n(a) \int_{-b}^b f_n(t, a) f_n(t, b) dt &= \frac{1}{\sqrt{2\pi}} \int_{-a}^a f_n(s, a) ds \int_{-b}^b f_n(t, b) e^{ist} dt \\ &= i^n \mu_n(b) \int_{-a}^a f_n(s, a) f_n(s, b) ds, \end{aligned}$$

by (1.3). Since  $f_n(t, a) f_n(t, b)$  is an even function of  $t$ , this can be rewritten as

$$(\mu_n(b) - \mu_n(a)) \int_{-a}^a f_n(s, a) f_n(s, b) ds = 2\mu_n(a) \int_a^b f_n(s, a) f_n(s, b) ds.$$

Divide by  $b - a$ , let  $b \rightarrow a + 0$ . By the continuity properties of  $f_n$  described in Theorem A and by (1.2),

$$\begin{aligned} \int_{-a}^a f_n(s, a) f_n(s, b) ds &\rightarrow \int_{-a}^a f_n^2(s, a) ds = 1 \\ \frac{1}{b-a} \int_a^b f_n(s, a) f_n(s, b) ds &\rightarrow f_n^2(a, a). \end{aligned}$$

Hence

$$\left( \frac{d\mu_n}{da} \right)^+ = \lim_{b \rightarrow a+0} \frac{\mu_n(b) - \mu_n(a)}{b - a} = 2\mu_n f_n^2(a, a).$$

Similarly, interchanging the roles of  $a$  and  $b$ ,

$$\left( \frac{d\mu_n}{da} \right)^- = \lim_{b \rightarrow a-0} \frac{\mu_n(b) - \mu_n(a)}{b - a} = 2\mu_n f_n^2(a, a).$$

That is to say that  $d\mu_n/da$  exists and

$$\frac{d\mu_n}{da} = 2\mu_n f_n^2(a, a).$$

The lemma follows on multiplying both sides by  $2\mu_n$  and remembering  $\lambda_n = \mu_n^2$ .

It is now easy to derive the behavior of  $\lambda_n$  as  $a \rightarrow \infty$  by integration of (1.12), provided a sufficiently accurate approximation to  $f_n(a, a)$  is known. The estimate (1.9) is not sufficient for this purpose.

## II. ASYMPTOTIC EXPRESSIONS FOR $f_n(t)$

An estimate of  $f_n^2(a)$  will be obtained by the comparison of two different asymptotic expressions for  $f_n(iv)$  ( $v \rightarrow \infty$ ), one derived from (1.3), the other from an application of the WKB method to the differential equation satisfied by  $f_n$ .

LEMMA 2. As  $v \rightarrow +\infty$

$$i^n \mu_n f_n(iv) = \pm \frac{1}{\sqrt{2\pi}} f_n(a) \frac{e^{av}}{v} + O(e^{av} v^{-2}). \quad (2.1)$$

PROOF. In (1.3) put  $t = iv$ . Two integrations by parts on the right hand side of the resulting equation combined with the fact that  $f_n(-a) = \pm f_n(a)$  yield the result.

The next lemma shows that the approximation (1.9) to  $f_n(t)$  is applicable to imaginary values of  $t$  in a certain range.

LEMMA 3.

$$f_n(t) = (2^n n! \pi^{1/2})^{-1/2} \phi_n(t) + o(1) \quad (2.2)$$

$$f'_n(t) = (2^n n! \pi^{1/2})^{-1/2} \phi'_n(t) + o(1), \quad (2.3)$$

uniformly in  $|t| < (\log a)^{1/3}$ . Here  $\phi_n(t)$  is defined in (1.11).

PROOF. Since  $f_n$  and  $\phi_n$  are either both odd or both even, there is a constant  $C = C_n(a)$  such that

$$y(t) = f_n(t) - C\phi_n(t)$$

satisfies

$$y(0) = y'(0) = 0.$$

By (1.9) and (1.10)

$$C = (2^n n! \pi^{1/2})^{-1/2} + O(a^{-1}) \quad (a \rightarrow \infty). \quad (2.4)$$

By means of Theorem A and of well-known recurrence relations for the Hermite polynomials it is easily verified that  $y(t)$  satisfies

$$\frac{d}{dt} \left[ (t^2 - a^2) \frac{dy}{dt} \right] + a^2 t^2 y = \chi_n y + r(t) \quad (2.5)$$

where

$$r(t) = \alpha \phi_{n-4}(t) + \beta \phi_n(t) + \gamma \phi_{n+4}(t) \quad (\phi_k \equiv 0 \text{ for } k < 0) \quad (2.6)$$

and

$$\alpha, \beta, \gamma = O(1) \quad (a \rightarrow \infty).$$

We shall prove Lemma 3 by using the method of successive approximations. Put

$$\begin{aligned} Y(t) &= \sup_{|z| \leq |t|} |y(z)| \\ m(t) &= \sup_{|z| \leq |t|} 2 |r(z)| \\ k(t) &= \sup_{|z| \leq |t|} 2a^{-2} |\chi_n - a^2 z^2|. \end{aligned}$$

All three functions are increasing functions of  $|t|$  and by (1.11) and (2.6)

$$m(t) < K |t|^{n+4} e^{|t|^2/2} \quad (|t| > 1). \quad (2.7)$$

By (1.6),

$$K |t|^2 < k(t) < K |t|^2 \quad (|t| > 1). \quad (2.8)$$

Integrate (2.5) along the straight line segment from 0 to  $u$ :

$$(u^2 - a^2) y'(u) = \int_0^u (\chi_n - a^2 v^2) y(v) dv + \int_0^u r(v) dv.$$

Hence, for

$$|u| < 2^{-1/2} a$$

$$|y'(u)| < 2a^{-2} \int_0^u |\chi_n - a^2 v^2| |y(v)| dv + 2a^{-2} \int_0^u |r(v)| dv,$$

$$|y'(u)| < k(u) \int_0^{|u|} Y(v) dv + a^{-2} m(u) |u|. \quad (2.9)$$

Hence, in

$$|t| < 2^{-1/2}a,$$

$$\begin{aligned} Y(t) &= \sup_{|z| \leq |t|} |y(z)| \\ &= \sup \left| \int_0^z y'(u) du \right| < k(t) \int_0^{|t|} du \int_0^{|u|} Y(v) dv + \frac{1}{2} a^{-2} m(t) |t|^2. \end{aligned} \quad (2.10)$$

As a first step we replace  $Y(v)$  in (2.10) by  $Y(t)$  and obtain

$$Y(t) < k(t) Y(t) \frac{|t|^2}{2!} + a^{-2} m(t) \frac{|t|^2}{2!}. \quad (2.11)$$

Next in (2.11) we replace  $t$  by  $v$  and use the resulting estimate of  $Y(v)$  in (2.10):

$$Y(t) < k^2(t) \frac{|t|^4}{4!} Y(t) + a^{-2} m(t) \left\{ \frac{|t|^2}{2!} + \frac{k(t)|t|^4}{4!} \right\}.$$

Returning again to (2.10) and integrating and continuing in this fashion leads to

$$Y(t) < \lim_{p \rightarrow \infty} \frac{k^p(t) |t|^{2p}}{(2p)!} Y(t) + a^{-2} m(t) \sum_{l=1}^{\infty} \frac{k^{l-1}(t) |t|^{2l}}{(2l)!}.$$

Therefore, using (2.7) and (2.8),

$$\begin{aligned} Y(t) &< a^{-2}(m(t)/k(t)) [\cosh(k^{1/2}(t)|t|) - 1] \\ &< Ka^{-2} |t|^{n+2} \exp\left\{\frac{1}{2}|t|^2 + K|t|^2\right\} \quad (1 < |t| < 2^{-1/2}a). \end{aligned}$$

Since  $Y(t)$  increases with  $|t|$ , we have in

$$|t| \leq (\log a)^{1/3} \quad (2.12)$$

$$|y(t)| \leq Y(t) < Ka^{-2} (\log a)^{(n+2)/3} \exp\{K(\log a)^{2/3}\}. \quad (2.13)$$

As  $a \rightarrow \infty$ , the right hand side of (2.13) tends to 0. Therefore, in the circle (2.12), by (2.4),

$$\begin{aligned} f_n(t) - (2^n n! \pi^{1/2})^{-1/2} \phi_n(t) &= (C - (2^n n! \pi^{1/2})^{-1/2}) \phi_n(t) + y(t) \\ &= O(a^{-1} (\log a)^{n/3} \exp\{\frac{1}{2}(\log a)^{2/3}\}) \\ &\quad + o(1) \\ &= o(1), \end{aligned} \quad (2.14)$$



which is (2.2). To obtain (2.3), use (2.13) in (2.9) to obtain  $y'(t) = o(1)$ , then differentiate (2.14).

LEMMA 4. As  $a \rightarrow \infty$ ,

$$\lambda_n \rightarrow 1.$$

PROOF. It is well known that

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} \phi_n(s) e^{ist} ds = i^n \phi_n(t).$$

Also, by (1.11), for real  $t$ ,

$$\begin{aligned} \int_{|s|>a} \phi_n(s) e^{ist} ds &= O\left(\int_{|s|>a} |s|^n e^{-s^2/2} ds\right) \\ &= o(1) \quad (a \rightarrow \infty). \end{aligned}$$

Substitute from (1.9) into (1.3):

$$\begin{aligned} i^n \mu_n \phi_n(t) &= (2\pi)^{-1/2} \int_{-a}^a \phi_n(s) e^{ist} ds + o(1) + o(|\mu_n|) \\ &= (2\pi)^{-1/2} \left( \int_{-\infty}^{\infty} - \int_{|s|>a} \right) + o(1) + o(|\mu_n|) \\ &= i^n \phi_n(t) + o(1) + o(|\mu_n|). \end{aligned}$$

Hence  $\mu_n \rightarrow 1$  and  $\lambda_n = \mu_n^2 \rightarrow 1$ .

LEMMA 5. For  $v \geq a^4$

$$f_n(iv) = i^n 2^{(3n+1)/2} \pi^{-1/4} (n!)^{-1/2} a^{n+1} e^{-a^2} v^{-1} e^{av} (1 + \eta),$$

where

$$\eta \rightarrow 0 \quad (a \rightarrow \infty),$$

uniformly in  $v \geq a^4$ .

PROOF. Let

$$w(v) = (a^2 + v^2)^{1/2} f_n(iv).$$

Then, expressing (1.5) in terms of  $w$ ,

$$w''(v) - \left\{ \frac{a^2}{(a^2 + v^2)^2} + \frac{a^2 v^2 + \chi_n}{a^2 + v^2} \right\} w(v) = 0. \quad (2.15)$$

Let

$$T = (\log a)^{1/4},$$

$$q^2(v) = \frac{a^2 v^2 + \chi_n}{a^2 + v^2},$$

$$s = \int_T^v q(t) dt,$$

$$z(s) = q^{1/2}(v) w(v).$$

In terms of  $z$  and  $s$  (2.15) becomes

$$\frac{d^2 z}{ds^2} - z(s) = \rho(s) z(s),$$

where

$$\rho(s) = \frac{1}{2} q''(v) q^{-3}(v) - \frac{3}{4} q'^2(v) q^{-4}(v) + a^2(a^2 + v^2)^{-2} q^{-2}(v).$$

Let

$$Z(s) = Ae^s + Be^{-s}, \quad (2.16)$$

where the constants  $A$  and  $B$  are determined by the conditions

$$Z(0) = z(0), \quad \frac{dZ}{ds}(0) = \frac{dz}{ds}(0).$$

By Lemma 3, (1.11) and (1.6)

$$\begin{aligned} z(0) &= q^{1/2}(T) (a^2 + T^2)^{1/2} f_n(iT) \\ &= (2i)^n (2^n n! \pi^{1/2})^{-1/2} a T^{n+1/2} e^{T^2/2} (1 + o(1)) \quad (a \rightarrow \infty), \\ \frac{dz}{ds}(0) &= (q^{1/2}(T) (a^2 + T^2)^{1/2} i f'_n(iv)) \Big/ \frac{ds}{dv} \Big|_{v=T} \\ &\quad + f_n(iT) \frac{d}{dv} [q^{1/2}(v) (a^2 + v^2)^{1/2}] \Big/ \frac{ds}{dv} \Big|_{v=T} \\ &= q^{-1/2}(T) (a^2 + T^2)^{1/2} i f'_n(iT) + \dots \\ &= (2i)^n (2^n n! \pi^{1/2})^{-1/2} a T^{n+1-1/2} e^{T^2/2} (1 + o(1)) \quad (a \rightarrow \infty), \end{aligned}$$

This yields

$$A = \frac{1}{2} (z(0) + z'(0)) = (2i)^n (2^n n! \pi^{1/2})^{-1/2} a T^{n+1/2} e^{T^2/2} (1 + o(1)), \quad (2.17)$$

$$B/A \rightarrow 0 \quad (a \rightarrow \infty). \quad (2.18)$$

We want to show that  $Z(s)$  can be used as an approximate value for  $z(s)$ . Let

$$D(s) = z(s) - Z(s).$$

Then

$$D(0) = D'(0) = 0$$

and

$$e^{-s} \frac{d}{ds} \left[ e^s \left( \frac{dD}{ds} - D \right) \right] = \frac{d^2 D}{ds^2} - D = \rho(s) [D(s) + Z(s)],$$

$$\begin{aligned} \frac{dD}{ds} - D(s) &= e^{-s} \int_0^s \rho(x) e^x [D(x) + Z(x)] dx \\ &= e^s \frac{d}{ds} (e^{-s} D(s)). \end{aligned}$$

Therefore

$$\begin{aligned} e^{-s} D(s) &= \int_0^s e^{-2y} dy \int_0^y \rho(x) e^x [D(x) + Z(x)] dx \\ &= \frac{1}{2} \int_0^s \rho(x) [D(x) + Z(x)] e^{-x} (1 - e^{-2(s-x)}) dx. \end{aligned} \quad (2.19)$$

If

$$\mu = \sup_{s>0} e^{-s} D(s)$$

then, by (2.16) and (2.19)

$$\mu < \frac{1}{2} \mu \int_0^\infty |\rho(x)| dx + \frac{1}{2} |A| \int_0^\infty |\rho(x)| dx + \frac{1}{2} |B| \int_0^\infty |\rho(x)| dx. \quad (2.20)$$

Now

$$\int_0^\infty |\rho(x)| dx = \int_T^\infty |\rho(s(t))| q(t) dt. \quad (2.21)$$

Straightforward differentiations give

$$\left| \frac{q'(t)}{q(t)} \right| < \frac{K}{t}, \quad \left| \frac{q''(t)}{q(t)} \right| < \frac{K}{t^2} \quad (t \geq T),$$

$$\begin{aligned} |\rho(s(t))| q(t) &< Kq^{-1}t^{-2} < K(a^2 + t^2)^{1/2} a^{-1}t^{-3}; \\ |\rho(s(t))| q(t) &< Kt^{-3} \\ |\rho(s(t))| q(t) &< Ka^{-1}t^{-2} \end{aligned} \quad \begin{aligned} (T \leq t \leq a) \\ (a < t). \end{aligned}$$

Therefore, by (2.21),

$$\int_0^\infty |\rho(x)| dx < KT^{-2},$$

and so, by (2.20), (2.17), and (2.18),

$$\mu < o(|A| + |B|) = o(|A|) \quad (a \rightarrow \infty).$$

By the definition of  $\mu$  this implies

$$\begin{aligned} z(s) &= Z(s) + o(|A| e^s) \quad (s > 0; a \rightarrow \infty), \\ &= Ae^s(1 + o(1)), \end{aligned} \quad (2.22)$$

by (2.16) and (2.18).

Now

$$\begin{aligned} s &= \int_T^v \left( \frac{a^2 t^2 + \chi_n}{a^2 + t^2} \right)^{1/2} dt \\ &= \int_T^v \frac{at}{(a^2 + t^2)^{1/2}} \left[ 1 + \frac{1}{2} \frac{\chi_n}{a^2 t^2} + O\left(\frac{\chi_n^2}{a^4 t^4}\right) \right] dt \\ &= [a(a^2 + t^2)^{1/2} + \frac{1}{2} \chi_n a^{-2} \log t(a + (a^2 + t^2)^{1/2})^{-1}]_T^v + o(1), \end{aligned}$$

uniformly in  $v$ . Hence, expanding in powers of  $a/v$  and of  $T/a$ , respectively

$$s = av - a^2 - \frac{1}{2} T^2 - (n + \frac{1}{2}) \log T + (n + \frac{1}{2}) \log(2a) + o(1) \quad (a \rightarrow \infty) \quad (2.23)$$

uniformly in

$$v \geq a^4.$$

By (2.22), (2.23), (2.17) in  $v \geq a^4$

$$\begin{aligned} f_n(iv) &= q^{-1/2}(v) (a^2 + v^2)^{-1/2} z(s) \\ &= a^{-1/2} v^{-1} (2i)^n (2^n n! \pi^{1/2})^{-1/2} a T^{n+1/2} e^{T^2/2} \left(\frac{2a}{T}\right)^{n+1/2} e^{av - a^2 - T^2/2} \\ &= i^n 2^{(3n+1)/2} (n! \pi^{1/2})^{-1/2} a^{n+1} e^{-a^2} v^{-1} e^{av} (1 + o(1)), \end{aligned}$$

which proves the lemma.

### III. COMPLETION OF THE PROOF OF THEOREM 1

By comparison of Lemma 2 and Lemma 5

$$\begin{aligned} \pm f_n(a) &= (2\pi)^{1/2} \cdot 2^{(3n+1)/2} \pi^{-1/4} (n!)^{-1/2} a^{n+1} e^{-a^2} \mu_n (1 + o(1)) \\ &= 2^{(3n+2)/2} \pi^{1/4} (n!)^{-1/2} a^{n+1} e^{-a^2} \mu_n (1 + o(1)) \end{aligned}$$

as  $a \rightarrow \infty$ .

Hence, by Lemma 1,

$$\begin{aligned} \mu_n^{-2} \cdot \lambda_n^{-1} \frac{d\lambda_n}{da} &= \lambda_n^{-2} \frac{d\lambda_n}{da} = 4\mu_n^{-2} f_n^2(a) \\ &= 4\pi^{1/2} (n!)^{-1} 8^n \cdot 4a^{2n+2} e^{-2a^2} (1 + o(1)). \end{aligned}$$

Integrating from  $a$  to  $\infty$  and using Lemma 4 and an integration by parts

$$\begin{aligned} \lambda_n^{-1} - 1 &= 4\pi^{1/2} (n!)^{-1} 8^n \int_a^\infty 4a^{2n+2} e^{-2a^2} da (1 + o(1)) \\ &= 4\pi^{1/2} (n!)^{-1} 8^n a^{2n+1} e^{-2a^2} (1 + o(1)). \end{aligned}$$

Theorem 1 follows after multiplication by  $\lambda_n$ .

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